

$$Z_t = \phi Z_{t-1} + W_t$$

Let  $\lambda_1, \lambda_2$  be eigenvalues of  $\phi$ ,  $c_1, c_2$  are eigenvectors

$$\Rightarrow \lambda_1 c_1 = \phi c_1, \quad \lambda_2 c_2 = \phi c_2$$

$$\Rightarrow \phi(c_1, c_2) = (c_1, c_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (\Rightarrow \text{partitioned matrix})$$

$$\Rightarrow \phi = (c_1, c_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (c_1, c_2)^{-1} \quad (\Rightarrow \text{Spectral decomposition})$$

$$= C \Lambda C^{-1} \quad \text{where } C = (c_1, c_2), \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (\text{if } \lambda_1 \neq \lambda_2)$$

rewrite VAR(1) as

$$Z_t = C \Lambda C^{-1} Z_{t-1} + W_t$$

$$\Rightarrow C^{-1} Z_t = \Lambda C^{-1} Z_{t-1} + C^{-1} W_t$$

$$\Rightarrow \eta_t = \Lambda \eta_{t-1} + \hat{W}_t \quad \text{where } C^{-1} W_t = \hat{W}_t, \quad C^{-1} Z_t = \eta_t, \quad Z_t = C \eta_t$$

$$\Rightarrow \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \eta_{1t-1} \\ \eta_{2t-1} \end{pmatrix} + \begin{pmatrix} \hat{W}_{1t} \\ \hat{W}_{2t} \end{pmatrix} \quad \begin{matrix} \Downarrow \\ \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix} \end{matrix}$$

$\Rightarrow Z$  is linear combination of  $\eta$

(case I):  $|\lambda_1| < 1, |\lambda_2| < 1 \Rightarrow \eta_{1t}$  is  $I(0), \eta_{2t}$  is  $I(0)$

$\Rightarrow \begin{matrix} Z_{1t} \\ Z_{2t} \end{matrix}$  is  $I(0), \begin{matrix} Z_{1t} \\ Z_{2t} \end{matrix}$  is  $I(0)$

$\Rightarrow Z_t$  is stationary

(case II)  $\lambda_1 = 1, |\lambda_2| < 1 \Rightarrow \eta_{1t}$  is  $I(1), \eta_{2t}$  is  $I(0)$

$\Rightarrow Z_{1t}$  is  $I(1), Z_{2t}$  is  $I(1)$

a linear combination of  $Z_{1t}$  and  $Z_{2t}$  is  $I(0)$

$\Rightarrow Z_t$  is cointegrated

$\eta_{1t}$  is  $I(2)$

$\eta_{1t}$  is  $I(1)$

(case III)  $\lambda_1 = 1, \lambda_2 = 1$

$$\begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_{1t-1} \\ \eta_{2t-1} \end{pmatrix} + \begin{pmatrix} \hat{W}_{1t} \\ \hat{W}_{2t} \end{pmatrix} \Rightarrow Z_t \text{ is } I(2)$$

Jordan form