Optional Math for FW Theorem

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In a graduate level econometrics course, FW theorem can be proved using matrix algebra. This note presents a proof without using matrix. As a starting point, recall the first order conditions (FOC) of OLS

$$\sum residual_i = 0 \tag{1}$$

$$\sum residual_i \times regressor_i = 0 \tag{2}$$

See eq (9), (10) on page 17 of this lecture notes `http://www.fsb.miamioh.edu/lij14/311r_slide_ch2n.pdf`. Now consider a multiple regression with two regressors:

$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \hat{\beta}_2 x_{2i} + \hat{u}_i, \qquad (i = 1, 2, \ldots n) \tag{3}$$

where beta hat is estimated coefficient, and u hat is residual. FW theorem suggests a two-step procedure, and we can write the step-one regression (regressing x1 onto x2) as

$$x_{1i} = \hat{\alpha}_0 + \hat{\alpha}_1 x_{2i} + \hat{r}_i \tag{4}$$

where r hat is residual in step-one regression. Next we multiply each term in the original regression (3) by r hat, and do summation. It follows that

$$\sum y_i \hat{r}_i = \sum \hat{\beta}_0 \hat{r}_i + \sum \hat{\beta}_1 x_{1i} \hat{r}_i + \sum \hat{\beta}_2 x_{2i} \hat{r}_i + \sum \hat{u}_i \hat{r}_i \tag{5}$$

Because r hat is residual, and x2 is the regressor in step-one regression, FOC (1) and (2) imply that

$$\sum \hat{\beta}_0 \hat{r}_i = \hat{\beta}_0 \sum \hat{r}_i = 0$$

$$\sum \hat{\beta}_2 x_{2i} \hat{r}_i = \hat{\beta}_2 \sum x_{2i} \hat{r}_i = 0$$

Moreover, since u hat is residual, and x1 and x2 are regressors in regression (3), we have

$$\sum \hat{u}_i \hat{r}_i = \sum \hat{u}_i (x_{1i} - \hat{\alpha}_0 - \hat{\alpha}_1 x_{2i}) = \sum \hat{u}_i x_{1i} - \hat{\alpha}_0 \sum \hat{u}_i - \hat{\alpha}_1 \sum \hat{u}_i x_{2i} = 0$$

In short, the first, third, and fourth terms on the right hand side of equation (5) are zero, and equation (5) can be simplified as

$$\sum y_i \hat{r}_i = \sum \hat{\beta}_1 x_{1i} \hat{r}_i = \hat{\beta}_1 \sum x_{1i} \hat{r}_i \tag{6}$$

and we can solve for $\hat{\beta}_1$

$$\hat{\beta}_1 = \frac{\sum y_i \hat{r}_i}{\sum x_{1i} \hat{r}_i} = \frac{\sum y_i \hat{r}_i}{\sum (\hat{\alpha}_0 + \hat{\alpha}_1 x_{2i} + \hat{r}_i) \hat{r}_i} = \frac{\sum y_i \hat{r}_i}{\hat{\alpha}_0 \sum \hat{r}_i + \hat{\alpha}_1 \sum x_{2i} \hat{r}_i + \sum \hat{r}_i^2} = \frac{\sum y_i \hat{r}_i}{\sum \hat{r}_i^2} \tag{7}$$

The finding that

$$\hat{\beta}_1 = \frac{\sum y_i \hat{r}_i}{\sum \hat{r}_i^2} \tag{8}$$

justifies the step-two regression—we obtain $\hat{\beta}_1$ by regressing y onto $\hat{r}$.

Takeaway of FW Theorem—only <u>part of</u> x_1 is useful for estimating or identifying β_1. Only the part of x_1 that cannot be explained by x_2 matters. Eat fish body, not whole fish!

A fundamental issue of econometrics is to <u>identify which part</u> of x_1 can be used to infer the causal effect of x_1 on y.

Run a multiple regression that controls for x_2, if we have data.

Eco 411 will show you that, amazingly, how to use <u>instrumental variable</u> to get the fish body part even if we have <u>no</u> data for x_2 (for example, school district, or beautifulness of a house).