Math Review: Exp and Log Functions

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1. Exponential and Logarithm functions play important roles in economics and statistics. This note will review them

2. The derivative of exp function is itself

$$\frac{de^x}{dx} = e^x \tag{1}$$

see `http://www.fsb.miamioh.edu/lij14/311r_or_derivative.pdf`. Consider a general exp function

$$y = e^{\beta x}, \qquad (\textit{exponential growth model}) \tag{2}$$

The <u>chain rule</u> implies that

$$\frac{dy}{dx} = e^{\beta x}\beta = y\beta \tag{3}$$

Let $dx = 1$. Then we can show $dy = y\beta$, or

$$\textit{growth rate} \equiv \frac{dy}{y} = \beta = \textit{constant}, \qquad (\textit{constant growth rate}) \tag{4}$$

In short, the exponential growth model indicates that the growth rate $\frac{dy}{y}$ is constant. By contrast, the linear growth model $y = \beta x$ indicates that the change is constant $dy = \beta = \textit{constant}$. If we believe, say, income grows by 2 percent (not 2k dollars) each year, then the exponential growth model below is appropriate

$$\textit{income} = \textit{initial income} \times e^{0.02 \textit{year}}, \quad (\textit{year} = 1, 2, ...)$$

3. After taking log of exponential growth model (2), we get log level model

$$\textit{log}(y) = \beta x, \qquad (\textit{log level model}) \tag{5}$$

This is an example of <u>generalized linear model</u> because the term on the right hand side βx is a linear function. The model is "generalized" since the model becomes linear only after the log transformation.

4. Log transformation is widely used because it can simplify multiplication into addition

$$\log(c_1 c_2) = \log(c_1) + \log(c_2) \tag{6}$$

   In old days when computers were not invented, this property can make math calculation much easier.

5. Another reason for its popularity is, if a variable follows a log-normal distribution, which is skewed, taking log can change it into symmetric normal distribution

$$Z \sim Log\ Normal\ Distribution \Rightarrow log(Z) \sim Normal\ Distribution \tag{7}$$

   For instance, the growth rate of income u_{year} may be random

$$income = initial\ income \times e^{\sum u_{year}}, \quad (year = 1, 2, ...)$$

   Then income follows log normal distribution

$$log(income) = log(initial\ income) + \sum u_{year} \sim Normal\ Distribution$$

   since $\sum u_{year}$ is normally distributed according to central limit theorem

6. Finally, log transformation can downplay the effect of outliers (extreme values), and make data more homogenous

$$log(10000) = 9.21034 < 10000$$