Eco311 Optional Reading: Magic 1.96 and Numerical Method

(Jing Li, Miami University)

1. The goal of this note is to reinforce students' understanding of the critical value 1.96 by applying the numerical method to evaluate a probability.

2. For most people, it is unclear how to compute the probability such as

$$P(-cr < Z < cr) = \texttt{special large probability number} \tag{1}$$

   where Z denotes a standard normal random variable with mean of zero and variance of one $Z \sim N(0,1)$. We want to focus on "special" "large" probability number like 0.95 in order to say something almost certain about the random variable[1] Z.

3. In particular, we want to find so called critical value denoted by cr in (1). The magic 1.96 is just cr associated with `special large probability number` = 0.95! So we are 95% sure that a standard normal random variable is between -1.96 and 1.96.

4. Equivalently, we can show $P(Z < 1.96) = 0.975$. That means if we have many many standard normal random values, 1.96 is greater than 97.5 percent of them. In other words, 1.96 is the 97.5-th percentile of standard normal distribution.

5. Recall that the standard normal random variable is continuous—it can take any real number like 1, 1.9, 1.96, 1.967..., you name it. The continuity also implies that the probability is given by an integral, other than the commonly used summation or $\sum$ notation (which applies to a discrete random variable)

6. Mathematically we have

$$P(-cr < Z < cr) = \int_{-cr}^{cr} \phi(z)dz = \int_{-cr}^{cr} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{2}$$

   (a) $\phi(z)$ is called the probability density function (pdf).

   (b) For standard normal distribution $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, which is symmetric around zero, and looks like a bell or hump.

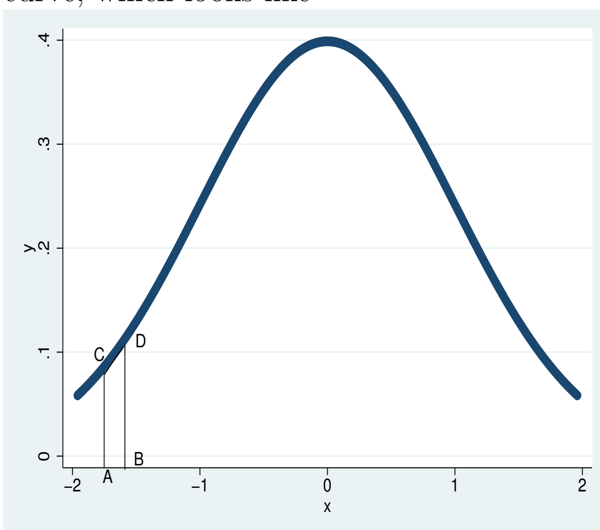
[1] I know my young daughter can be moody and unpredictable. But I am almost sure that she will be happy after playing ipad.

I know this kind of math looks depressive. Especially you may wonder how come there are rocket-science numbers of π and e. Well, it takes two geniuses—Laplace and Gauss to figure out those math, so you can wait until pursuing a PhD.

7. But I do want to show you the intuition behind the math. Let me slow down:

   (a) We know the total probability must equal one: $P(-\infty < Z < \infty) = 1$

   (b) By construction, the total probability is equal to the total area under the pdf curve, which looks like

   Clearly, the standard normal pdf is symmetric and bell-shaped.

   (c) To compute the total area, we can divide it into many small areas, and each small area can be approximated by a trapezoid such as ACDB shown in the above graph.

   (d) The integral is nothing but the limit of sum of all those trapezoids as AB goes to zero. You can know more about integration by reading any textbook of calculus. Remember, integration is essentially summation.

8. To put together everything

$$
\begin{aligned}
1 &= P(-\infty < Z < \infty) && (3)\\
&= \texttt{Total area under pdf curve} && (4)\\
&\approx \texttt{Many small trapezoids} && (5)\\
&= \lim_{AB\to 0} \sum_i ACDB_i && (6)\\
&= \lim_{AB\to 0} \sum \left(\frac{ACDB}{AB}\right) AB && (7)\\
&= \lim_{surface\to 0} \sum \left(\frac{\texttt{mass}}{\texttt{surface}}\right) \texttt{surface} && (8)\\
&= \lim_{surface\to 0} \sum (\texttt{density})\, \texttt{surface} && (9)\\
&= \int_{-\infty}^{\infty} \phi(z) dz && (10)
\end{aligned}
$$

(a) Equation (6) divides the total area into many trapezoids.

(b) Equation (8) treats the i-th trapezoid $ACDB_i$ as a heavy object with mass, which is placed on the surface AB.

(c) Loosely speaking we may define density as the ratio of mass to surface, and equation (9) shows that.

(d) Finally, we use $\phi(z)$ to denote the probability density, use dz to denote the surface, and replace the $\lim \sum$ with the integration sign $\int$.

9. Now you understand that the height of pdf is <u>proportional to</u> the probability. In fact the pdf function does not return probability; it returns probability density. That is why the pdf can be greater than one (but probability cannot). We see greater-than-one pdf when a lot of probability mass is concentrated in small volume, that is, when the standard deviation gets smaller.

10. The trapezoid approximation (6) suggests a numerical method (an iterative procedure) to compute $P(-1.96 < Z < 1.96)$

(a) We let $A = -1.96, B = A + 0.01 = -1.95$. Note AB $= 0.01$ is very small. The integral in theory requires $AB \to 0$.

(b) We compute $C = \phi(A) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(-1.96)^2}{2}}$, $D = \phi(B) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(-1.95)^2}{2}}$. R has a built-in function `dnorm` doing this.

(c) The trapezoid area ABDC (probability mass) is given by

$$\frac{AB(AC+DB)}{2} = \frac{0.01\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{(-1.96)^2}{2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{(-1.95)^2}{2}}\right)}{2} \tag{11}$$

(d) Next, we update (move to the right by 0.01), let $A = -1.95, B = -1.94$, and compute a new trapezoid area.

(e) We continue moving to right and computing a new trapezoid area until $B = 1.96$ (doing a loop or iteration). Then the probability $P(-1.96 < Z < 1.96)$ can be approximated by the total sum of all those trapezoid areas.

11. R codes are below

```
x = seq(-1.96, 1.96, 0.01)
height = dnorm(x)
plot(x, height)

area = 0
for (i in 1:(length(x)-1)) {
trapezoid = (height[i]+height[i+1])*0.01/2
area = area + trapezoid
}
area

> area
[1] 0.9500023
```

The approximation answer 0.9500023 is very close to the true value 0.95!

12. Numerical method is a very powerful tool. In this case, it is the easiest way to evaluate the probability of standard normal distribution. This is because the antiderivative of ϕ has no close-form. In other words, we cannot fund a function $f(\,)$ so that its derivative is $\phi(\,)$. We can only compute $P(-1.96 < Z < 1.96)$ numerically, not analytically[2].

[2]There are other approaches such as finding the antiderivative of the Taylor expansion of $\phi(z)$. To learn more, google "error function".

13. To summarize, now you know where the numbers in Table G.1 (page 743, sixth edition of Wooldridge's book) come from.
14. Exercise: can you modify my code to show $P(-1.645 < Z < 1.645) = 0.9$? where 1.645 is another magic number (critical value).