“Trend-Cycle Decomposition Allowing for Multiple Smooth Structural Changes in the Trend of US Real GDP”

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Abstract

A key feature of Flexible Fourier Form (FFF) is that the essential characteristics of multiple structural breaks can be captured using a small number of low frequency components from a Fourier approximation. We introduce a variant of the FFF into the trend function of US real GDP in order to allow for gradual effects of unknown numbers of structural breaks occurring at unknown dates. We find that the hypothesis of no breaks can be rejected, and the Fourier components are significant. Our new cycle matches the NBER chronology very well, especially for the Great Recession of 2009.

Keywords: Trend-Cycle Decomposition; Flexible Fourier Form; Smooth Trend Breaks

JEL Classification: E32, E37, C32

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1. Introduction

The appropriate way to decompose real US GDP into its trend and cyclical components has received a substantial amount of attention in the macroeconomics literature. Beveridge and Nelson (1981) define the trend as the limiting forecast as the horizon goes to infinity, adjusted for the mean rate of growth. As such, trend growth is a random walk plus drift and the cycle is the difference between the actual value of the series and its estimated trend. An identifying restriction in the Beveridge and Nelson (BN) decomposition is that the correlation coefficient between innovations in the trend and the cycle is -1. As is well-known, for the real US GDP series, this restriction produces a stochastic trend that is quite volatile and a cycle that is small.

The unobserved components (UC) approach of Clark (1987) decomposes GDP such that trend is a random walk plus drift and the cycle is a stationary AR(2) process. Harvey and Jaeger (1993) extend the UC model by allowing the cycle to depend on a cosine wave, and Kim and Nelson (1999) allow the cycle to be a Markov switching process. Unlike the BN decomposition, the UC trend is smooth and the UC cycle is large and highly persistent. This difference is mainly due to the fact that the UC model imposes the restriction that innovations in the trend and cycle are uncorrelated. In a very important paper, Morley et al. (2003) (hereafter MNZ) introduce an unrestricted trend-cycle correlation into the UC model and find that it yields the same decomposition as the BN method.

If the trend is misspecified, the cycle is necessarily misspecified as well. The extensions of the BN and UC models have focused on the cycle with relatively little effort going into the specification of the trend. A notable exception is Perron and Wada (2009) (hereafter PW). PW provide evidence that it is important to account for a structural break in the trend of real US GDP occurring in 1973Q1. Once allowing for a break in the drift term, the estimated variance of the trend innovation is close to zero and the estimated cycle corresponds very closely to the NBER chronology. Their point is that previous studies misspecified the trend
by ignoring the productivity reduction resulting from the oil price shock of 1973. A similar
argument is made by Basistha (2007) for the Canadian economy.

However, PW use a single dummy variable to represent the break. The implicit assump-
tions are (i) the break date is known to be 1973Q1, or the break is exogenous; (ii) the break
has abrupt or instantaneous effect on the trend, and (iii) the number of breaks is only one.
In our view all three assumptions are restrictive. First, assumption (i) seems ad hoc given
the fact the OPEC increased the oil price by 5.7% on April 1, 11.9% on June 1, 17% on
October 16, and declared a complete export embargo on October 20. It is unclear which
of these dates is the true break date. For assumption (ii), even if the price jumps are best
modeled as being sharp, the effects are likely to be gradual as it took time for the price
increases to manifest themselves in output reductions. Finally, the assumption of a single
break may also be suspect in that a number of studies suggest that the reduction in growth
trend actually began sometime in the late 1960s or very early 1970s. For instance, see Bai
et al. (1998) and the Symposium on the Slowdown in Productivity Growth in the Journal of
Economic Perspectives of 1988. Moreover, Basu et al. (2001) find that there was a resump-
tion of productivity growth in the 1990s that is suggestive of another break in the trend.
More recently, the effects of the 2009 financial crisis on GDP are certainly indicative of a
new structural break. The point is that a model allowing for an unknown number of possibly
smooth breaks is likely to be superior to a model that contains only a single sharp break.

In this paper, we allow for gradual effects of unknown numbers of structural breaks
occurring at unknown dates in the GDP trend. Specifically, we introduce a variant of the
Flexible Fourier Form (FFF) of Gallant (1981) into the trend function. It has been well
demonstrated by Becker et al. (2004), Becker et al. (2006) and Enders and Lee (2012) that
the essential characteristics of one or more unknown structural breaks can be captured using a
small number of low frequency components from a Fourier approximation. One nice property
of the FFF is that we do not need to assume that the break dates or the number of breaks
are known \( a \text{ priori} \).

The second advantage of the FFF is that it allows for the gradual effects of structural breaks. Other nonlinear models such as the Logistic Smooth Transition Autoregression (LSTAR) can capture smooth breaks as well. But like the dummy-variable approach, the LSTAR requires that the number of breaks be known. Finally, estimating the FFF is much easier than the LSTAR model because the FFF model is linear in the parameters for given frequency.

The rest of this paper is organized as follows. Section 2 provides a description of using the FFF to approximate structural breaks. Section 3 presents the FFF trend-cycle decomposition. Section 4 discusses estimation results and compares the decomposition from our model to those from other models. Section 5 concludes.

2. Approximating Structural Breaks with the Fourier Form

Consider a time-series with the intercept term \( \alpha(t) \), which is time-varying because of structural breaks. Usually the number and the location of the breaks are unknown. Instead of searching for those breaks, we consider a Fourier approximation for \( \alpha(t) \) given as:

\[
\alpha(t) \approx \mu + \sum_{k=1}^{n} \alpha_k \sin(2\pi kt/T) + \sum_{k=1}^{n} \beta_k \cos(2\pi kt/T), \quad (n \leq T/2)
\]

where \( n \) represents the number of frequencies, \( k \) represents the index for frequencies, and \( T \) is the number of observations. Note that the traditional model of a constant intercept is nested within (1). In the absence of structural breaks, we should not be able to reject the null hypothesis that all values of \( \alpha_k \) and \( \beta_k \) equal 0.

The Fourier series is capable of approximating absolutely integrable functions to any desired degree of accuracy. Beginning with \( n = 1 \), it is always possible to improve the approximation by using additional frequencies. When \( n = T/2 \) is reached, the fit will
be perfect. Instead of selecting break dates, the specification problem is transformed into incorporating the appropriate frequency components into the estimating equation. The immediate benefit of using the Fourier approximation is that it is not necessary to assume that the break dates or the number of breaks are known \textit{a priori}.

As shown in Becker et al. (2004), Becker et al. (2006) and Enders and Lee (2012), one or more structural breaks can be captured using a small number of low frequency components from the Fourier approximation. This is so since breaks tend to shift the spectral density function towards the frequency of zero. Because the FFF tries to approximate the breaks, it can be thought of as a semiparametric model. Remarkably, the FFF provides a global, rather than a local, approximation. Other approximations, such as Taylor series, are valid only at a particular point in the sample space. Moreover, unlike the polynomial function, the trigonometric function used by the Fourier form are always bounded.

The FFF can also approximate the break in the slope of a stochastic trend. Consider a random walk with drift

\[ y_t = \alpha(t) + y_{t-1} + \epsilon_t \]  

where \( \alpha(t) \) can still be approximated by (1), but now \( \alpha(t) \) is interpreted as the drift term. The time-varying drift term results in the time-varying slope of the trend.

By using simulation, Panels 1 to 4 of Figure 1 illustrate the ability of the FFF to mimic the behavior of smooth breaks in a trending series. The solid lines in the four panels show the simulated trend with which the shift in the intercept term has the smooth transition representation. Panel 1 uses the LSTAR representation with parameter values \( d_1 = 2, \gamma = 0.05, T = 500, \delta = 0.015, \) and \( \lambda = 0.5 \) :

\[ y_t = d_1/[1 + \exp(\gamma(t - \lambda T))] + \delta t \]  

Here the smooth break occurs in the middle of sample, and the intercept term declines
gradually over time: at $t = 0$, 250, and 500, the intercept is approximately $d_1$, $d_1/2$, and 0, respectively. Panel 2 shows the Exponential Smooth Transition Autoregressive (ESTAR) break with parameter values $d_1 = 2$, $\gamma = 0.0002$, $T = 500$, $\delta = 0.01$, and $\lambda = 0.75$:

$$y_t = d_1[1 - \exp(-\gamma(t - \lambda T)^2)] + \delta t$$

(4)

Now the break happens in $3T/4$, and unlike the LSTAR model, the ESTAR model allows the intercept to return to its initial value. Panel 3 shows the trend with two LSTAR breaks in the intercept:

$$y_t = d_1/[1 + \exp(\gamma_1(t - \lambda_1 T))] + d_2/[1 + \exp(\gamma_2(t - \lambda_2 T))] + \delta t$$

(5)

where $d_1 = 2$, $d_2 = 1$, $\gamma_1 = 0.05$, $\gamma_2 = -0.05$, $\delta = 0.015$, $\lambda_1 = 0.2$ and $\lambda_2 = 0.75$. The two breaks have different magnitudes and (partially) offset each other. Finally in Panel 4 the second break occurs in a new location, and reinforces the first break. The parameter values are $d_1 = 2$, $d_2 = 1$, $\gamma_1 = 0.05$, $\gamma_2 = 0.05$, $\delta = 0.015$, $\lambda_1 = 0.2$ and $\lambda_2 = 0.67$. In each panel, the short-dashed line and the long-dashed line represent the fitted values (denoted by 1-Frequency and 2-Frequencies, respectively) from the below FFF regressions estimated by the Ordinary Least Squares (OLS):

$$\hat{y}_t = \hat{\mu} + \hat{\alpha}_1 \sin(2\pi t/T) + \hat{\beta}_1 \cos(2\pi t/T) + \hat{\delta} t$$

(6)

$$\hat{y}_t = \hat{\mu} + \hat{\alpha}_1 \sin(2\pi t/T) + \hat{\beta}_1 \cos(2\pi t/T) + \hat{\alpha}_2 \sin(4\pi t/T) + \hat{\beta}_2 \cos(4\pi t/T) + \hat{\delta} t$$

(7)

Although these types of smooth breaks might be expected to be seen in economic variables with a slowly evolving trend, there is no simple way to incorporate LSTAR or ESTAR trend functions in the UC framework without knowing the number of breaks. Even if under rare circumstances the number of breaks is known, estimating the LSTAR or ESTAR function is
much harder than the FFF because the STAR function is nonlinear in the parameters (for instance, $\gamma$ and $\lambda$ in the LSTAR function are notorious for being hard to identify).

Despite the simplicity of the FFF, we can see from Figure 1 that a Fourier approximation using the single frequency $n = 1$ can mimic the breaks with fairly high degrees of accuracy. The centered $R^2$ of Equation (6) is between 0.909 and 0.996. It is obvious that the second frequency component ($n = 2$) improves the fit. The improvement is especially noticeable at the extremes of the sample. This is important because typically our interest is on the most recent history (the end of the sample). In Section 4 we will revisit this issue. In short, Figure 1 indicates that (i) one or two frequency components is able to capture much of the variation in the trend; and (ii) the FFF approximation works well irrespective of the location and the form of breaks.

Although the Fourier series is especially suitable for smooth breaks, we also illustrate sharp breaks in Figure 2, in which the sample size is only 100, and both the intercept and the slope of the trend are allowed to change. The solid lines in Panels 1 to 4 show four trends with the types of sharp breaks used in Becker et al. (2006). Panel 1 shows a sharp break in the intercept at $T/2$. Panel 2 shows a sharp change in the intercept and slope at $2T/3$. Panel 3 illustrates two declines in the slope of a continuous trend, and Panel 4 shows one decline and one increase in the slope. As in Figure 1, the short-dashed line depicts the Fourier approximation using the single frequency $n = 1$ and the long-dashed line depicts the approximation using two frequencies $n = 2$. Overall, Figure 2 corroborates the key points shown by Figure 1. A small number of low frequency components ($n = 1$ or $n = 2$) is sufficient to capture the essential features of different types of breaks.

Using additional frequency seems to be particularly important if the breaks are very sharp (see Panel 2). Note that the FFF approximation works very well in Panel 3 and 4 of Figure 2. These are the cases if we model the GDP trend as a random walk with time-varying drift. Events such as Oil Crisis shifts the drift term, which in turn causes the slope of the trend
to change.

3. The Model

Our new trend-cycle decomposition employs the same set-up as Morley et al. (2003) except that the FFF (trigonometric terms) are included in the trend:

\[ y_t = \tau_t + c_t \]  
\[ \tau_t = \mu + \text{FFF} + \tau_{t-1} + \eta_t \]  
\[ \text{FFF} \equiv \sum_{k=1}^{n} a_k \sin(2\pi kt/T) + \sum_{k=1}^{n} b_k \cos(2\pi kt/T) \]  
\[ (1 - \phi_1 L - \phi_2 L^2)c_t = e_t \]  
\[ \begin{pmatrix} \eta_t \\ e_t \end{pmatrix} \sim \text{i.i.d.} N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\eta^2 & \sigma_{\eta e} \\ \sigma_{\eta e} & \sigma_e^2 \end{pmatrix} \right) \]

where \( y_t \) is 100 times the log quarterly real US GDP, \( \tau_t \) is the unobserved trend, and \( c_t \) is the cycle. Following the literature (for instance, see Clark (1987) and Morley et al. (2003)) we let the cyclical component takes the AR(2) form (11).

The trend and cycle innovations (\( \eta_t \) and \( e_t \)) are jointly normally distributed with mean zero and a general variance-covariance matrix. We follow Morley et al. (2003) and do not restrict the covariance \( \sigma_{\eta e} \) to be zero. One important issue is comparing the standard errors \( \sigma_\eta \) and \( \sigma_e \), which measure the volatility of the shocks in trend and cycle, respectively. Because the trend \( \tau_t \) is a random walk with drift, the trend shock \( \eta_t \) has permanent effect, and is commonly referred to as the real shock.

As mentioned in Introduction, the cycle will be ill-estimated if the trend is misspecified. In particular, this paper tries to account for possible structural breaks in the trend. Toward that end, \( \mu + \text{FFF} \) in (9) denotes the possibly time-varying drift term in the trend. If at least one break is present and the break causes the drift term to change, then some or all
trigonometric terms in the FFF (10) should be statistically significant. On the other hand, if $a_k = 0, b_k = 0$ for all $k$ in (10), then our model is reduced to the unrestricted UC (URUC) model used by Morley et al. (2003). Put differently, the URUC model assumes the drift term is a constant $\mu$ (and so there is no break) and our model nests the URUC model as a special case.

It is easy to see the link between our model and the PW model of Perron and Wada (2009). The PW model utilizes a dummy variable to represent the break of 1973 Oil Crisis, and the dummy variable equals zero before 1973Q1 and equals one thereafter. Mathematically, the PW model replaces (10) with

\[
\begin{align*}
\text{d} \ast \text{break dummy, break dummy} &= \begin{cases} 
0, & \text{before 1973Q1;} \\
1, & \text{after 1973Q1.}
\end{cases}
\end{align*}
\]  

(13)

where $d$ is the coefficient of the dummy variable for the break. The PW model implies a slowdown in GDP growth if the estimated $d$ is negative.

Unlike the PW model, we do not assume the number and location of the breaks are known. Instead, we let data talk, and try different values of $n$, the number of frequencies used by the FFF. As a practical matter, it is not desirable to use a large value of $n$. The use of many frequencies can exhaust degrees of freedom and can lead to over-fitting. Here we follow the principle of parsimony and the recommendation of Gallant and Souza (1991), and try $n = 1$, $n = 2$ and $n = 3$ in (10). For each given $n$, the test for the null hypothesis $H_0 : a_k = b_k = 0, \forall k$, follows a standard distribution. Rejecting the null hypothesis is equivalent to rejecting the use of a constant drift term in the trend specification.

Notice that the FFF (10) is linear in the parameters $a_k$ and $b_k$. As a result, our model is much easier to estimate than including the LSTAR or ESTAR function in the trend function (9). Thus we can say our model is a computationally tractable way of allowing for gradual

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1The model is unrestricted in the sense that it does not restrict $\sigma_{\nu}$ to be zero.
effects of the structural breaks. Finally, our model is cast in state space form and estimated by the Kalman smoother. The Kalman smoother uses all information available in the sample, while the basic Kalman filter only uses information available up to time t. Thus, the Kalman smoother provides superior fit compared to the basic Kalman filter.

4. Estimation Results

4.1 Using MNZ Short Sample

We download the US real GDP from Federal Reserve Economic Data (FRED)\(^2\). The same sample period of Morley et al. (2003), which is from 1947Q1 to 1998Q2, is used here for ease of comparison. Table 1 reports the results of Maximum Likelihood estimations\(^3\) of all models\(^4\). The estimates of the URUC model are close to those reported in Morley et al. (2003). Note that the estimated standard deviation of the trend innovation \(\sigma_\eta\) is highly significant. The trend and cycle innovations are negatively correlated and their correlation coefficient is close to BN decomposition. In contrast, \(\sigma_\eta\) is insignificant and \(\sigma_{\eta e}\) is positive in the PW model, mainly because the PW model accounts for the break in 1973Q1 and the coefficient of the break dummy \(d\) is negative and significant.

We try three specifications of the FFF. In Table 1, Fourier 1, 2, 3 denote our model (9) with \(n = 1, 2, 3\) in the FFF (10), respectively. Note that the Fourier model and PW model both indicate insignificant \(\sigma_\eta\), and both imply positive correlation (\(\sigma_{\eta e}\)) of the trend and cycle innovations. Despite these similarities, the Fourier model differs from the PW model in the way of modeling breaks. Note that \(a_1\), the coefficient of the first trigonometric term, is significant in all three Fourier models. The Akaike Information Criterion (AIC) suggests

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\(^2\)The website is [http://research.stlouisfed.org/fred2/data/GDPC1.txt](http://research.stlouisfed.org/fred2/data/GDPC1.txt).

\(^3\)The RATS codes are available upon request.

\(^4\)The Hodrick-Prescott (HP) filter can also be used to approximate a smoothly evolving nonlinear trend. In addition to the well-known disadvantages of the HP filter, it does not readily allow for testing the null hypothesis of linearity (no breaks). Therefore in this paper we do not consider the HP filter.
that the Fourier model with three frequencies (Fourier 3) has the best fit, in which three out of six trigonometric terms ($a_1, a_3$ and $b_3$) are significant. This can be seen as the first evidence that at least one break is present in the trend.

One econometric issue is to determine whether it is possible to reject the null hypothesis of no breaks in the GDP trend. We consider the likelihood ratio (LR) statistic:

$$LR = -2(\log L_{\text{restricted}} - \log L_{\text{unrestricted}}),$$

(14)

where $\log L_{\text{restricted}}$ is the log likelihood of the URUC model, which is restricted in this context because it imposes the null hypothesis of no breaks. The unrestricted models are the PW model and our Fourier models. Under the null hypothesis the LR test follows a Chi-squared distribution with degrees of freedom equal to the number of restrictions\(^5\). Table 1 shows that the hypothesis of no breaks is rejected at the 5\% level in all cases.

We also report the Bayesian Information Criterion (BIC). A well known fact is that BIC tends to pick a model that is more parsimonious than AIC. Among the three Fourier models, BIC picks the Fourier 1 model. Thus, it is debatable which Fourier model is the best one, depending on which criterion to use. But one thing is for sure. The LR test indicates undoubtedly that there is at least one break in the GDP trend. In this regard, our results complement Perron and Wada (2009) by considering the possibility of smooth breaks and multiple breaks.

Next we compare the trend-cycle decomposition obtained from the Fourier 1 model (selected by BIC) to those obtained from the URUC and PW models. Panel 1 of Figure 3 compares the Fourier 1 trend to URUC trend. Notice that the Fourier 1 trend suggests a multiplicity of breaks. Relative to the URUC trend, the shape of the Fourier 1 trend indi-

\(^5\)The Chi-squared distribution is valid because we compute the LR test for given frequency and for given break dummy. In other words, here we do not use the sup Wald test suggested by Quandt (1960), which follows a nonstandard distribution.
icates that the productivity growth was strong (the trend was convex) during middle 60’s, and weak (the trend was concave) in late 70s and early 80s. The resumption of productivity growth in 90s is not obvious in the Fourier 1 trend.

Panel 2 compares the Fourier 1 trend to PW trend. We find the two trends largely overlap. However, since the PW trend uses only one sharp break at 1973Q1, it is piecewise linear, indicating that productivity growth was constant during 1955-1972 and 1973-1998, respectively. Therefore, the PW trend cannot capture other possible breaks.

For the sake of completeness, Panels 3 and 4 draw the Fourier 3 trend (selected by AIC). The difference between Fourier 3 trend and URUC trend is similar to Panel 1. But the difference between the Fourier 3 trend and PW trend in Panel 4 is more obvious than Panel 2. For instance, the failure of the PW trend to capture the break in early 60s is clearly demonstrated in Panel 4. Moreover, Fourier 3 trend indicates that the slowdown of GDP growth may start in late 60s, and this finding is consistent with Bai et al. (1998).

Panels 5-8 draw the cycle components along with the recessions dated by NBER (shaded). It is shown that the Fourier cycle is generally smaller in magnitude than the PW cycle and URUC cycle. The implication about the severity of recession is also different. Take the recession in 1982. The Fourier 1 cycle indicates the 1982 recession is less severe than that implied by the PW cycle and URUC cycle. The gap between the Fourier 3 cycle and PW cycle is most apparent in 60s. This result is expected since in that period there may be a break but the PW model rules out the possibility of its existence.

4.2 Using the Recent Data

Updating the results is particularly interesting because the MNZ short sample does not include the recessions of 2001 and 2009. As such, we reestimate all models using the extended sample that covers the 1947Q1-2013Q3 period. Table 2 reports the new results. Once again, the hypothesis of no breaks in the trend is rejected in all cases. Moreover, the PW model
and Fourier models indicate insignificant $\sigma_\eta$ and positive $\sigma_{\eta e}$.

Nevertheless, including the 2001 and 2009 recessions in the estimation leads to several noticeable changes. First, in the URUC model $\sigma_\eta$ decreases from 1.06 to 0.44, and $\sigma_{\eta e}$ now becomes positive and insignificant. Second, in the PW model the coefficient of the break dummy falls from -0.20 to -0.26. That means the 2001 and 2009 recessions cause further slowdown in GDP growth, but the PW model mistakenly attributes it to the 1973 Oil crisis. Third, for the Fourier models, AIC picks the Fourier 2 as the best model. BIC still chooses the Fourier 1 model.

The effects of the recent two recessions can be best seen in Figure 4. In Panel 1, for instance, those two recessions effectively pull the URUC trend down. As a result, the Fourier 1 trend is almost always above URUC trend. That translates to the Fourier 1 cycle being consistently below URUC cycle in Panel 5. In our view, the Fourier 1 cycle seems more reasonable than the URUC cycle because the former appears “mean-reverting” while the latter shows persistent deviation from the value of zero.

Furthermore, the gap between the Fourier 1 cycle and PW cycle in Panel 6 is wider than the Panel 6 in Figure 3. So using additional recessions is really helpful for identification purpose. With more recessions, the results of the PW model and Fourier models become more distinguishable.

The Great Recession is especially important in that it can be used as a benchmark to evaluate the relevance of all models. According to the NBER chronology the Great Recession ended in 2009Q1. That turning point is captured only by the Fourier 2 cycle shown in Panels 7 and 8. In contrast, the URUC cycle and PW cycle both fail to predict the end of the Great Recession. Actually the URUC cycle and PW cycle imply that the recession would have worsen even after 2010. Therefore, in terms of matching with the latest reality, the Fourier 2 model outperforms all other models.

To check the robustness and ensure the fairness in comparison, we estimate a new PW
model with two dummy variables: the first one still represents the 1973Q1 break, and the second one uses 2008Q3 as the break date. The estimated coefficients of the two dummy variables are -0.19 and -0.68 respectively, and both are significant at the 5% level. The coefficient of the second dummy variable confirms that the Great Recession causes further slowdown in the GDP growth. Figure 5 compares the Fourier 2 cycle to the cycle obtained in the new PW model (denoted by PW2). Note that the PW2 cycle is able to capture the turning point where Great Recession ended. This in turn implies that capturing that turning point requires taking the Great Recession into account. The Fourier 2 model of course does that, but the URUC and old PW models fail to do so.

Noticeable difference between the Fourier 2 cycle and PW2 cycle still exists, for example, in the period 1965-1969. This fact, along with the finding of Bai et al. (1998), casts doubt on using 1973Q1 as the break date. If the GDP slowdown really started in late 60s other than in 1973, then the PW2 cycle would overestimate the true cycle.

Another striking difference can be seen in the end of sample: the PW2 cycle indicates a strong and rapid recovery after the Great Recession. According to the PW2 cycle, the US GDP would return to its potential level in 2012. The Fourier 2 cycle, however, predicts a much weaker and slower recovery, which is in line with the projection of Congressional Budget Office that US GDP would not reach its potential level until 2017. In light of this, the Fourier model outperforms the new PW model even though the new PW model has taken into account the Great Recession.

In this case the superiority of Fourier model may be attributed to the momentum or lingering effect of the Great Recession. Just because a recession ends does not necessarily mean its effect will disappear instantaneously. The Fourier model involves trigonometric

7There is no qualitative change in $\sigma_q$, $\sigma_e$ and $\sigma_{ge}$ in the new PW model relative to the old PW model. Full estimation results of the new PW model are available.
8The report can be found at http://www.cbo.gov/publication/43907.
functions, and therefore is suitable for capturing the smooth effect of a break.

**Conclusion**

This paper provides a new trend-cycle decomposition of US real GDP. The focus is on accounting for possibly multiple breaks in the trend component, and the breaks may have gradual effects. It would be hard, if not impossible, to achieve those two goals by using the traditional methods. For example, Perron and Wada (2009) account for only one “exogenous” break using the dummy-variable approach. The estimation would become exponentially more difficult if one attempted to allow for multiple “endogenous” breaks using a grid search for the unknown break dates. The same is true if we try to allow for gradual (smooth) breaks using the conventional nonlinear models such as LSTAR and ESTAR. For one thing, just like the dummy-variable approach, those models necessitate knowing or estimating the break dates. It would soon become computationally intractable if we directly augment the trend component with the LSTAR or ESTAR function.

Our idea of using the Flexible Fourier Form (FFF) to approximate multiple breaks provides a simple solution. We have shown, both through simulation and real-data estimation, that FFF can successfully approximate various types of breaks, regardless of their number and locations. The FFF has two distinguished features. First, it provides a global rather than local approximation for the trend with breaks. Second, the FFF is linear in the unknown parameters for given frequency, and that makes FFF much easier to estimate than the LSTAR and ESTAR models. We have shown a small number of low frequencies are sufficient to capture very complicated pattern in breaks.

After applying the FFF decomposition to the recent sample of 1947Q1-2013Q3, we find that (i) the hypothesis of no breaks in GDP trend can be rejected; (ii) the standard deviation of the trend innovation is insignificant; (iii) the innovations of trend and cycle are positively correlated (hence, instead of shocks causing the trend and cycle to move in opposite di-
rections, our model finds that negative shocks result in simultaneous declines in both the trend and the cycle); (iv) the FFF cycle matches the NBER dating very well, especially for the Great Recession; (v) the new PW model that accounts for both 1973 and 2008 breaks implies a rapid recovery after the Great Recession, whereas our Fourier model indicates a slow recovery.

We expect that the FFF methodology can be applied in other settings whenever the break, or nonlinearity in general, need to be accounted for. One potential topic is following Hillebrand (2005) but using the FFF to approximate the breaks in the conditional volatility of financial time series. The dummy-variable approach of Bai and Perron (1998) imposes an upper bound for the number of breaks, which may be too restrictive for a high-frequency financial time series with many observations. In that case the FFF approach can be less restrictive and computationally more feasible.
References


Table 1: Estimation Results, Short Sample (1947Q1-1998Q2)

<table>
<thead>
<tr>
<th></th>
<th>URUC</th>
<th>PW</th>
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<th>Fourier 2</th>
<th>Fourier 3</th>
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<td>s.e.</td>
<td>Est</td>
<td>s.e.</td>
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<td>0.96*</td>
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<td>0.43*</td>
<td>(0.13)</td>
<td>0.44*</td>
</tr>
<tr>
<td>$\sigma_{\eta e}$</td>
<td>-0.95*</td>
<td>(0.14)</td>
<td>0.36*</td>
<td>(0.07)</td>
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</tr>
<tr>
<td>$\phi_1$</td>
<td>1.34*</td>
<td>(0.09)</td>
<td>1.51*</td>
<td>(0.08)</td>
<td>1.51*</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>-0.40*</td>
<td>(0.06)</td>
<td>-0.58*</td>
<td>(0.08)</td>
<td>-0.58*</td>
</tr>
<tr>
<td>$a_1$</td>
<td>0.13*</td>
<td>(0.03)</td>
<td>0.13*</td>
<td>(0.03)</td>
<td>0.13*</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.02</td>
<td>(0.04)</td>
<td>0.01</td>
<td>(0.04)</td>
<td>0.06</td>
</tr>
<tr>
<td>$a_2$</td>
<td>-0.06</td>
<td>(0.05)</td>
<td>-0.06</td>
<td>(0.03)</td>
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</tr>
<tr>
<td>$b_2$</td>
<td>-0.00</td>
<td>(0.06)</td>
<td>0.04</td>
<td>(0.04)</td>
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<tr>
<td>$a_3$</td>
<td>0.15*</td>
<td>(0.05)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_3$</td>
<td>0.18*</td>
<td>(0.06)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d$</td>
<td>-0.20*</td>
<td>(0.04)</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$\log L$</td>
<td>-267.3</td>
<td>-261.0</td>
<td>-261.4</td>
<td>-260.8</td>
<td>-254.4</td>
</tr>
<tr>
<td>LR</td>
<td>12.5*</td>
<td>11.7*</td>
<td>12.9*</td>
<td>25.8*</td>
<td></td>
</tr>
<tr>
<td>AIC</td>
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<td>2.60</td>
<td>2.62</td>
<td>2.63</td>
<td>2.59</td>
</tr>
<tr>
<td>BIC</td>
<td>2.75</td>
<td>2.72</td>
<td>2.75</td>
<td>2.79</td>
<td>2.78</td>
</tr>
</tbody>
</table>

Note:

a: * denotes significance at the 5% level.
b: URUC model denotes the unrestricted unobserved component model used by Morley et al. (2003); PW model is the model used by Perron and Wada (2009); Fourier 1, 2, 3 denote our new model (9) with $n = 1, 2, 3$ in the FFF (10) respectively.
c: the sample is 1947Q1-1998Q2. The GDP data are from http://research.stlouisfed.org/fred2/data/GDPC1.txt.
Table 2: Estimation Results, Extended Sample (1947Q1-2013Q3)

<table>
<thead>
<tr>
<th></th>
<th>URUC</th>
<th></th>
<th>PW</th>
<th></th>
<th>Fourier 1</th>
<th></th>
<th>Fourier 2</th>
<th></th>
<th>Fourier 3</th>
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<td>Est</td>
<td>s.e.</td>
<td>Est</td>
<td>s.e.</td>
<td>Est</td>
<td>s.e.</td>
<td>Est</td>
<td>s.e.</td>
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<td>s.e.</td>
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<tr>
<td>( \mu )</td>
<td>0.82* (0.05)</td>
<td>0.97* (0.04)</td>
<td>0.80* (0.02)</td>
<td>0.78* (0.02)</td>
<td>0.79* (0.02)</td>
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<td></td>
</tr>
<tr>
<td>( \sigma_\eta )</td>
<td>0.44* (0.03)</td>
<td>0.00 (0.21)</td>
<td>0.00 (0.21)</td>
<td>0.00 (0.10)</td>
<td>0.00 (0.10)</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>( \sigma_\varepsilon )</td>
<td>0.65* (0.20)</td>
<td>0.48* (0.21)</td>
<td>0.46* (0.13)</td>
<td>0.33 (0.19)</td>
<td>0.36 (0.25)</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>( \sigma_{\eta \varepsilon} )</td>
<td>0.07 (0.12)</td>
<td>0.28* (0.13)</td>
<td>0.30* (0.08)</td>
<td>0.35* (0.08)</td>
<td>0.34* (0.12)</td>
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</tr>
<tr>
<td>( \phi_1 )</td>
<td>1.52* (0.11)</td>
<td>1.51* (0.10)</td>
<td>1.54* (0.09)</td>
<td>1.57* (0.09)</td>
<td>1.56* (0.07)</td>
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</tr>
<tr>
<td>( \phi_2 )</td>
<td>-0.53* (0.11)</td>
<td>-0.55* (0.09)</td>
<td>-0.57* (0.09)</td>
<td>-0.63* (0.09)</td>
<td>-0.62* (0.07)</td>
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</tr>
<tr>
<td>( a_1 )</td>
<td>0.15* (0.04)</td>
<td>0.14* (0.02)</td>
<td>0.14* (0.02)</td>
<td>0.14* (0.02)</td>
<td>0.14* (0.02)</td>
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</tr>
<tr>
<td>( b_1 )</td>
<td>-0.03 (0.05)</td>
<td>-0.07 (0.04)</td>
<td>-0.06 (0.04)</td>
<td>-0.06 (0.04)</td>
<td>-0.06 (0.04)</td>
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</tr>
<tr>
<td>( a_2 )</td>
<td>0.11* (0.04)</td>
<td>0.11* (0.03)</td>
<td>0.11* (0.03)</td>
<td>0.11* (0.03)</td>
<td>0.11* (0.03)</td>
<td></td>
<td></td>
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<tr>
<td>( b_2 )</td>
<td>-0.15* (0.05)</td>
<td>-0.15* (0.05)</td>
<td>-0.15* (0.05)</td>
<td>-0.15* (0.05)</td>
<td>-0.15* (0.05)</td>
<td></td>
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</tr>
<tr>
<td>( a_3 )</td>
<td>0.07 (0.06)</td>
<td>0.07 (0.06)</td>
<td>0.07 (0.06)</td>
<td>0.07 (0.06)</td>
<td>0.07 (0.06)</td>
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<tr>
<td>( b_3 )</td>
<td>0.03 (0.06)</td>
<td>0.03 (0.06)</td>
<td>0.03 (0.06)</td>
<td>0.03 (0.06)</td>
<td>0.03 (0.06)</td>
<td></td>
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</tr>
<tr>
<td>( d )</td>
<td>-0.26* (0.06)</td>
<td>-0.26* (0.06)</td>
<td>-0.26* (0.06)</td>
<td>-0.26* (0.06)</td>
<td>-0.26* (0.06)</td>
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</tr>
<tr>
<td>( \log L )</td>
<td>-333.9</td>
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<td>LR Test</td>
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<td>7.28*</td>
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<td>18.80*</td>
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<td>AIC</td>
<td>2.55</td>
<td>2.52</td>
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<tr>
<td>BIC</td>
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- b: the sample is 1947Q1-2013Q3. The GDP data are from http://research.stlouisfed.org/fred2/data/GDPC1.txt.
Figure 1: Fourier Form Approximation of Smooth Breaks

Panel 1: One LSTAR Break at T/2

Panel 2: One ESTAR Break at 3T/4

Panel 3: Two Offsetting LSTAR Breaks at T/5 and 3T/4

Panel 4: Two Reinforcing LSTAR Breaks at T/5 and 2T/3
Figure 2: Fourier Form Approximation of Sharp Breaks

Panel 1: Decline in Intercept at T/2

Panel 2: Decline in Intercept and Slope at 2T/3

Panel 3: Two Declines in Slope at T/3 and 2T/3

Panel 4: One Decline and One Increase in Slope at T/5 and 3T/4
Figure 3: Trend and Cycle, Short Sample (1947Q1-1998Q2)

Panel 1: Fourier 1 Trend vs URUC Trend

Panel 2: Fourier 1 Trend vs PW Trend

Panel 3: Fourier 3 Trend vs URUC Trend

Panel 4: Fourier 3 Trend vs PW Trend

Panel 5: Fourier 1 Cycle vs URUC Cycle

Panel 6: Fourier 1 Cycle vs PW Cycle

Panel 7: Fourier 3 Cycle vs URUC Cycle

Panel 8: Fourier 3 Cycle vs PW Cycle
Figure 4: Trend and Cycle, Extended Sample (1947Q1-2013Q3)
Figure 5: Fourier 2 Cycle vs PW2 Cycle

Year
Deviation (%)
-8 -6 -4 -2 0 2 4 6

CYCLE_PW2 CYCLE_F2

![Graph showing deviation over time for Fourier 2 Cycle vs PW2 Cycle.](image-url)